NASA TECHNICAL NOTE

NASA TN D-5259

c. 1

NASA TN D-5259

AN ANALYSIS OF SATELLITE POSITIONAL UNCERTAINTY BY STATISTICAL MECHANICS

by N. L. Bonavito

Goddard Space Flight Center
Greenbelt, Md.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JULY 1969

AN ANALYSIS OF SATELLITE POSITIONAL UNCERTAINTY BY STATISTICAL MECHANICS

By N. L. Bonavito

Goddard Space Flight Center
Greenbelt, Md.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

ABSTRACT

This paper treats, from a statistical mechanics viewpoint, the justification for acknowledging satellite positional uncertainty through in-track error buildup. In addition, by relating the angular momentum to a quantity defined as the virial parameter, through a derived quantity defined as the Vector Density function, it is shown that for any conservative system, the periodic variations, and hence the cross track error, will time average to zero. This is a direct consequence of the conservation of angular momentum.

SUMMARY

The present paper treats, from a statistical mechanics viewpoint, the justification for acknowledgment of satellite positional uncertainty through in-track error buildup. The equations of motion of the Hamilton-Jacobi theory specify a region or volume in phase space in which the system point moves. Since the system is in statistical equilibrium, it will initially have a specified energy, and the theorem of the integral invariants of Poincaré guarantees that volume in phase space in which the system point moves with an initially specified energy will be constant in time, although forces influencing the system will cause a distortion or elongation of the volume with time. Therefore, in traversing this distorted volume, an error factor in the equations of transformation will necessitate a change in the constant of the motion which relates to the time of perigee passage and, hence, the mean anomaly. Because the secular part of the mean anomaly is by far the largest portion, any error accumulated here can be brought out in terms of an in-track projection and will be mainly secular in nature. In addition, by *relating the angular momentum to a quantity defined as the virial parameter, through a derived quantity defined as the Vector Density Function,* it is shown that for any conservative system the periodic variations (of the orbit), and hence the cross-track error, will time average to zero. *This is a direct consequence of the conservation of angular momentum.*

CONTENTS

AN ANALYSIS OF SATELLITE POSITIONAL UNCERTAINTY BY STATISTICAL MECHANICS

by

N. L. Bonavito
Goddard Space Flight Center

INTRODUCTION

If we consider a $2n$-dimensional Cartesian space formed of coordinates $q_1 \cdots q_n, p_1 \cdots p_n$, and known as phase space, the complete dynamical specification of a mechanical system will be given by a point in such a space. The values of q and p at any time t can be obtained from their initial values by a canonical transformation that is a continuous function of time; i.e., there exists a canonical transformation from the values of the coordinates and momenta at any time t to their initial values. Obtaining this transformation is equivalent to solving the problem of the system motion. While the exact motion of any system is completely determined in classical mechanics by the initial conditions, in practice these conditions are often only incompletely known. In addition, the presence of nonconservative forces and uncertainties in fundamental data introduces further errors into the solution for the motion of such a system. By introducing certain techniques of statistical mechanics, one can make predictions about certain average properties by examining the motion of a large number of identical systems. The values of the desired quantities are computed by forming averages over all systems in the ensemble. Statistical techniques can be justified by showing that the time average of any quantity pertaining to any single system of interest would actually agree with the ensemble average for that quantity calculated for all members of the corresponding ensemble. The postulate leading to this conclusion is the ergodic hypothesis that states that the phase point for any isolated system will pass in succession through every point before finally returning to its original position in phase space.

STATEMENT OF THE PROBLEM

Consider the motion of the system point in phase space for the case of an artificial earth satellite. For such points, Liouville's Theorem states that the density of systems in the neighborhood of some given system in phase space remains constant in time. That is,

$$\frac{dD}{dt} = \frac{\partial D}{\partial t} + \sum_{i=1}^{3N} \left(\frac{\partial D}{\partial p_i} \dot{p}_i + \frac{\partial D}{\partial q_i} \dot{q}_i \right) ,$$

or,

$$\frac{dD}{dt} = \frac{\partial D}{\partial t} + [D,H] = 0 ,$$

where D is the density, H is the Hamiltonian, N is the number of particles, and p_i and q_i are the generalized or canonical momenta and position coordinates, respectively. These are given by

$$\left[p_i, H\right] = -\frac{\partial H}{\partial q_i} = \dot{p}_i ,$$

and

$$\left[q_i, H\right] = +\frac{\partial H}{\partial p_i} = \dot{q}_i .$$

When the ensemble of systems is in statistical equilibrium, the number of systems in a given state must be constant in time. Then $[D,H] = 0$, or the Poisson bracket vanishes. Therefore, we have equilibrium by choosing the density to be a function of the constants of the motion of the system. For conservative systems, D can be any function of the energy. The density is a constant equal to zero for the system with a given energy considered here.

Now canonical transformations are defined as having the property of preserving the form of Hamilton's equations of motion under transformation. There are other expressions invariant under canonical transformation. The theorem of Poincaré states that the integral

$$J_1 : \iint_s \sum_i dq_i \, dp_i$$

is invariant under canonical transformation. The proof of invariance can be reduced to showing that the sum of the Jacobians is invariant (Reference 1). In a similar fashion, the integral invariants can be extended to

$$J_n = \int \cdots \int dq_1 \cdots dq_n \, dp_1 \cdots dp_n ,$$

where the integral can be extended over an arbitrary region in phase space. The invariance of J_n is equivalent to the statement that volume in phase space is invariant under canonical transformation. From this it follows that the volume in phase space is constant in time. Now the motion of a system point is simply a particular contact transformation of the canonical coordinates. Consequently the change in time of a region of phase space can be represented by a contact or canonical transformation; that is, the contact transformation determines motion within the region of constant volume. From the Hamilton-Jacobi theory, the motion of an artificial earth satellite for a conservative system can be described by

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1},$$

$$\beta_2 = \frac{\partial W}{\partial \alpha_2},$$

and

$$\beta_3 = \frac{\partial W}{\partial \alpha_3},$$

where α_1, α_2, and α_3 are related to the total energy, total angular momentum, and polar component of angular momentum, respectively; β_1, β_2, and β_3 are related to the time of perigee passage, argument of perigee, and geocentric right ascension, respectively; and W is Hamilton's characteristic function (References 2 and 3).

These equations of motion, then, specify a region or volume in phase space in which the system point moves. In the course of time, the volume may take on different shapes, but as long as the system remains a conservative one the volume remains constant. If now any of the earth's geodetic constants are incorrectly specified or ignored in the representation of the gravitational potential

$$V = -\frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{r_e}{r}\right)^n P_n^m(\cos\theta)\left(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\right)\right],$$

where μ is the product of the gravitational constant and the planet's mass, r_e is its equatorial radius, r and θ are the planetocentric distance and latitude of a field point, $C_{n,m}$ and $S_{n,m}$ are the coefficients of potential, λ is the angle of east longitude, and $P_n^m(\cos\theta)$ is the associated Legendre functions, then the path of the system point in phase space will deviate more and more in time from one governed by the exact representation of the potential given above. In addition the initial volumes described will differ, and each of these will again evolve separately in time. The system point for this orbit which is bounded and not closed will therefore not arrive at the place and time in phase space determined by the correct solution. This error is manifest principally in the equation of motion

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1},$$

and indicates that the time of arrival of the satellite at perigee is changed. With each revolution, the difference in the times of perigee passage increases, and the "position" of the system point in phase space becomes more removed from that of the exact solution. Therefore, there is a stretching or elongation of system point positions in phase space, corresponding to a time and position error for the satellite. Since the time of perigee passage is related to the mean anomaly by the relation $\hat{M} = \hat{n}\,(t + \beta_1)$, where $\hat{n}$ is the mean motion, then the error is accumulated directly by the mean anomaly, a fundamental orbital element. It can be shown that the secular part of the mean anomaly is at a minimum, on the order of J_2 times the periodic part, where J_2 is the coefficient of the second zonal harmonic of the earth's gravitational potential (Reference 2).

ANGULAR MOMENTUM AND THE VECTOR DENSITY FUNCTION

Referred to a fixed origin, the angular momentum of a particle of mass m is given by $\vec{L} = (\vec{r} \times \vec{p})$, where the momentum $\vec{p} = m\vec{v}$, and $\vec{r}$ is the position vector of the mass. For unit mass, this is given in component form by

$$\vec{L} = (\vec{r} \times \vec{v}) = \vec{i}\left(r_y v_z - r_z v_y\right) + \vec{j}\left(r_z v_x - r_x v_z\right) + \vec{k}\left(r_x v_y - r_y v_x\right) ,$$

with respect to a set of Cartesian axes. Let us write the virial parameter $G_{\beta\gamma} = r_\beta v_\gamma$, where β and γ (which equal 1, 2, or 3 exclusively), refer to a particular component x, y, or z. Now let us define a Levi-Civita type Vector Density $\vec{\delta}_{\alpha\beta\gamma}$, where α, β, and γ are 1, 2, or 3 exclusively. $\vec{\delta}$ is the unit vector $\vec{i}$, $\vec{j}$, or $\vec{k}$, according to whether α = 1, 2, or 3, respectively; positive or negative for $\alpha\beta\gamma$ an even or odd permutation of 123; and zero if any two subscripts are equal. For example, for $\alpha\beta\gamma = 123$, $\vec{\delta}_{123} = +\vec{i}$; if $\alpha\beta\gamma = 213$, $\vec{\delta}_{213} = -\vec{j}$. Using this, the angular momentum becomes

$$\vec{L} = \vec{i}\left[G_{yz} - G_{zy}\right] + \vec{j}\left[G_{zx} - G_{xz}\right] + \vec{k}\left[G_{xy} - G_{yx}\right] ,$$

or simply,

$$\vec{L} = \sum_{\alpha,\beta,\gamma=1}^{3} \vec{\delta}_{\alpha\beta\gamma}\, G_{\beta\gamma} .$$

Therefore, the angular momentum for the system can be expressed as individual sums of the virial function through the use of the vector density, $\vec{\delta}_{\alpha\beta\gamma}$.

POSITIONAL UNCERTAINTY

From the above, it is important to consider how any error in orbital elements, especially the very sensitive mean anomaly, is reflected in the satellite's positional uncertainly. Then the in-track position error can be defined as

$$\Delta T = \frac{\vec{v} \cdot \Delta\vec{r}}{|\vec{v}|} ,$$

where $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$ and $\vec{v} = \vec{i}\dot{x} + \vec{j}\dot{y} + \vec{k}\dot{z}$. Thus the in-track error is the error in position given by $\Delta\vec{r}$, projected along the unit tangent velocity vector. In addition the error normal to the orbit plane is given as

$$\Delta N = \frac{\vec{L} \cdot \Delta\vec{r}}{|\vec{L}|} ,$$

where

$$\vec{L} = (\vec{r} \times \vec{v}) .$$

Here the normal or cross-track error is the error in position $\Delta\vec{r}$, projected in the direction of the unit angular momentum vector. Also $|\vec{L}|$ is the magnitude of the constant angular momentum of the orbit. If now the time average of these quantities is examined, then for the cross-track error

$$\frac{1}{\tau}\int_0^\tau \frac{d}{dt}\Delta N dt = \frac{1}{\tau}\int_0^\tau \frac{d}{dt}\frac{\vec{L} \cdot \Delta\vec{r}}{|\vec{L}|} dt ,$$

where τ is some time interval. Since $\vec{L}$ is a constant of the motion, we can write

$$\frac{1}{\tau|\vec{L}|}\int_0^\tau \frac{d}{dt}(\vec{L} \cdot \Delta\vec{r})\, dt = \frac{1}{\tau|\vec{L}|}\int_0^\tau \frac{d}{dt}\left(\left[\sum_{\alpha,\beta,\gamma=1}^{3} \vec{\delta}_{\alpha\beta\gamma} G_{\beta\gamma}\right] \cdot \Delta\vec{r}\right) dt ,$$

$$= \frac{1}{\tau|\vec{L}|}\left\{\left([G_{yz} - G_{zy}]\Delta x\right) + \left([G_{zx} - G_{xz}]\Delta y\right) + \left([G_{xy} - G_{yx}]\Delta z\right)\right\}_0^\tau ,$$

$$= \frac{1}{\tau|\vec{L}|}\Big\{\Big(\left[G_{yz}(\tau)\Delta x(\tau) - G_{yz}(0)\Delta x(0)\right] - \left[G_{zy}(\tau)\Delta x(\tau) - G_{zy}(0)\Delta x(0)\right]\Big)$$

$$+ \Big(\left[G_{zx}(\tau)\Delta y(\tau) - G_{zx}(0)\Delta y(0)\right] - \left[G_{xz}(\tau)\Delta y(\tau) - G_{xz}(0)\Delta y(0)\right]\Big)$$

$$+ \Big(\left[G_{xy}(\tau)\Delta z(\tau) - G_{xy}(0)\Delta z(0)\right] - \left[G_{yx}(\tau)\Delta z(\tau) - G_{yx}(0)\Delta z(0)\right]\Big)\Big\} .$$

Now it can be shown mathematically that Δx, Δy, and Δz are functions of the changes in the six elements of the orbit, and for an elliptic orbit the variation in these elements will be complex periodic functions of the argument of latitude, true anomaly, eccentric anomaly, and inclination, and will hence be bounded (see appendix). Therefore, Δx, Δy, and Δz will exhibit this behavior and also be bounded. This is fairly obvious since the difference in Cartesian components between two ellipses will not be infinite. In addition, since the coordinates and velocities of the satellite remain finite, there is an upper bound to $G_{\beta\gamma}$. Then by choosing τ sufficiently long, each of the products,

$$\frac{1}{\tau}\left(G_{\beta\gamma}\,\Delta r_{\alpha}\right),$$

where $\Delta r_{\alpha} = \Delta x$, Δy, or Δz for $\alpha = 1, 2,$ or 3, respectively, and $\alpha \neq \beta \neq \gamma$, can be made as small as desired. Therefore, in this limit the time average for the error normal to the orbit with an angular momentum that is a constant of the motion is said to vanish.

For the in-track position error, since ΔT does not involve $\vec{L}$ or products of the form $r_{\beta}\, v_{\gamma}$ but rather components of the velocity and the magnitude of velocity, the above arguments do not apply; hence,

$$\frac{1}{\tau}\int_0^{\tau} \frac{d}{dt}\,\Delta T\, dt \neq 0 .$$

Therefore, errors of the normal will be periodic in nature with both short and long periodic effects present, at times causing distortion of the error curve. The in-track error curve will then be secular-like and can have periodic effects added. It should become larger than the cross-track error and grow without bound (Reference 4). This result seems reasonable in view of the fact that the major portion of the error buildup appears in the time of perigee passage, or mean anomaly, whose secular part is much larger than the periodic part.

REMARKS

The effects of the various terms of the associated Legendre functions (Reference 4) are briefly summarized here.

1. For the zonals, the effect is either secular or periodic, depending on whether n is even or odd.
2. The sectorial harmonics $(n = m)$ always produce secular-like changes.
3. For the Tesseral harmonics $(n \neq m)$, the $C_{n,m}$ terms induce a large periodic change with a secular-like effect added, and the $S_{n,m}$ terms will induce a secular effect with a periodic variation added.

Now, if nonconservative forces of a quasistatic nature are present, that is, if they draw energy very slowly, then the above discussions will apply to a great extent. For example, if a small amount of energy is withdrawn from the system only at the point of perigee passage, the above results are applicable after the orbital constants have been redetermined. In this way, the angular momentum will be a constant of the motion for a given revolution. If the total energy of the orbit is being constantly changed, then all time-average arguments do not apply, but the statistical mechanical considerations are still valid. Consequently, for a nonconservative system, as soon as the nonconservative forces are removed, the above time-average argument applies again. Since an exact classical description for the effects of these nonconservative forces is not available, then from a statistical mechanical standpoint the discrepancy between the trajectories of the phase points of the actual and computed system will be even greater. Presently, therefore, a reasonable determination of the positional uncertainty of a satellite can be made providing the change in the time of perigee passage can be accurately determined.

In addition, for conservative systems, as a consequence of the conservation of angular momentum, periodic variations in the orbit, and therefore the cross-track position error, will time average to zero. The secular changes and hence the in-track position error will grow steadily. This error is reflected principally in the time of perigee passage or the mean anomaly.

While the purpose of this paper has been to study the physical meaning of effects caused by the errors from gravity models, nonconservative forces, etc., on the satellite trajectories, the problem now reduces to that of determining, if possible, an accurate estimate or error bound for each of these sources and the resulting magnitude of this error bound projected through the mean anomaly and in-track position. This will constitute the subject matter for a future investigation.

ACKNOWLEDGMENTS

The author is especially grateful to Dr. John A. O'Keefe of Goddard Space Flight Center, and Dr. David John O'Keeffe of the Naval Ordnance Laboratory, White Oak, Maryland, for their valuable discussions that contributed to this report. The author is also indebted to Dr. Abolghassem Ghaffari and Mr. Harvey Walden of Goddard Space Flight Center for their careful checking of this report.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, March 7, 1969
311-07-21-01-51

REFERENCES

1. Tolman, Richard C., "Principles of Statistical Mechanics," London: Oxford University Press, 1961.

2. Vinti, J. P., "Theory of an Accurate Intermediary Orbit for Satellite Astronomy," J. Res. Nat. Bur. Std. (U.S.) 65 B(3), July to September 1961.

3. Bonavito, N. L., "Computational Procedure for Vinti's Theory of an Accurate Intermediary Orbit," NASA Technical Note D-1177, March 1962.

4. Ouellette, G., and Sconzo, P., "Correlation of Gravimetric and Satellite Geodetic Data," IBM Report No. CES-D181, September 11, 1967 to February 29, 1968.

Appendix

Discussion of Oscillatory Nature of Error in Satellite Coordinates

It was pointed out in the section on positional uncertainty that the error bound in the position of a satellite could be shown to be a complicated oscillatory function of several orbital elements. To do this, it should be assumed that any errors or uncertainty in the potential model of the earth give rise to a "force" which would then drive the computed orbit away from the true one. One can then use the variation of elements technique as given by Plummer to describe the effect of this force on the orbital elements and consequently the Cartesian coordinates.*

A more direct and less complicated approach is the following. Denote the spherical coordinates of the satellite as r the radius, θ the colatitude, and ϕ the geocentric right ascension. Now the potential energy of the system is given by the standard solution of Laplace's equation, or $V = V(r, \theta, \phi)$. The kinetic energy can be so written as

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2\right) .$$

If the Lagrangian L is not an explicit function of time, then

$$\frac{\partial L}{\partial t} = 0 = \frac{\partial H}{\partial t} ,$$

where the Hamiltonian

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L ,$$

the sum taken over all canonical coordinates. Thus the Hamiltonian is a constant of the motion. In addition, if the forces present are derivable from a conservative potential, then $H = E$, the total energy. Assuming that the potential is a function of r and θ, then, since the coordinate ϕ is cyclic, its canonically conjugate momenta p_ϕ = constant, or

$$P_\phi = mr^2 \sin^2 \theta \dot{\phi} .$$

*H. C. Plummer, "An Introductory Treatise on Dynamical Astronomy." New York: Dover Publications, 1960.

The Hamiltonian is

$$H = E = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2 \dot{\theta}^2 + \frac{1}{2} mr^2 \sin^2 \theta \dot{\phi}^2 + V(r,\theta) ,$$

$$= \frac{1}{2} m\dot{r}^2 + \text{`V'}(r,\theta) ,$$

where 'V' (r,θ) is an effective potential.

For stable motion along the radial direction,

$$\frac{\partial \text{`V'}(r,\theta)}{\partial r} = mr\dot{\theta}^2 + \frac{\partial V(r,\theta)}{\partial r} - \frac{P_\phi^2}{mr^3 \sin^2 \theta} ,$$

and

$$\frac{\partial^2 \text{`V'}(r,\theta)}{\partial^2 r} - m\dot{\theta}^2 + \frac{\partial^2 V(r,\theta)}{\partial r^2} + \frac{3P_\phi^2}{mr^4 \sin^2 \theta} > 0 ,$$

i.e., this is a positive quantity, since $V(r,\theta)$ goes as inverse powers of r, and therefore a stable point in the radius vector given by r_0 exists. Denoting

$$\frac{\partial^2 \text{`V'}(r,\theta)}{\partial r^2} = k ,$$

then, for a small perturbation, the radius vector will oscillate about the equilibrium position with a frequency given by

$$\omega_r = \sqrt{\frac{k}{m}} = \left(\dot{\theta}^2 + \frac{1}{m} \frac{\partial^2 V(r,\theta)}{\partial r^2} + \frac{3P_\phi^2}{m^2 r^4 \sin^2 \theta} \right)^{1/2} .$$

For the colatitude, let us assume a small change in angle θ only so that

$$H = E = \frac{1}{2} mr^2 \dot{\theta}^2 + \text{`}\hat{V}\text{'}(r,\theta) ,$$

where

$$\text{`}\hat{V}\text{'}(r,\theta) = V(r,\theta) + \frac{P_\phi^2}{2mr^2 \sin^2 \theta} ,$$

and

$$\frac{1}{2} mr^2 \dot{\theta}^2 = E - \text{`}\hat{V}\text{'}(r,\theta) .$$

If there were no displacement in θ, the relation between P_ϕ and the true colatitude, say $\theta = \theta_0$, would be given by

$$\left(\frac{\partial \text{`}\hat{V}\text{'}(r,\theta)}{\partial\theta}\right)_{\theta_0} = \left(\frac{\partial V(r,\theta)}{\partial\theta}\right)_{\theta_0} - \frac{P_\phi^2 \cos\theta_0}{mr^2 \sin^3\theta_0} .$$

Substituting P_ϕ,

$$\dot{\phi}^2 = \frac{1}{mr^2 \sin\theta_0 \cos\theta_0} \left(\frac{\partial V(r,\theta)}{\partial\theta}\right)_{\theta_0} .$$

Then for $\dot{\theta} = 0$, the expression for the total energy before displacement becomes

$$E_0 = V(r,\theta_0) + \frac{\sin\theta_0}{2\cos\theta_0} \left(\frac{\partial V(r,\theta)}{\partial\theta}\right)_{\theta_0} .$$

For an energy slightly larger than E_0, and angular momentum given by P_ϕ, the angle θ will oscillate about the value θ_0. Now setting

$$k = \left(\frac{\partial^2 \text{`}V\text{'}(r,\theta)}{\partial\theta^2}\right)_{\theta_0} = \left(\frac{\partial^2 V(r,\theta)}{\partial\theta^2}\right)_{\theta_0} + \frac{\left(\sin^2\theta_0 + 3\cos^2\theta_0\right)}{\sin\theta_0 \cos\theta_0} \left(\frac{\partial V(r,\theta)}{\partial\theta}\right)_{\theta_0} .$$

for small values of the displacement $\theta - \theta_0$, `$\hat{V}$' (r,θ) can be expanded in a Taylor series such that

$$\text{`}\hat{V}\text{'}(r,\theta) = E_0 + \frac{1}{2} k (\theta - \theta_0)^2 ,$$

and the expression for the energy becomes

$$E - E_0 = \frac{1}{2} mr^2 \dot{\theta}^2 + \frac{1}{2} k \left(\theta - \theta_0\right)^2 .$$

This is in the form of the energy for a harmonic oscillator with coordinate $(\theta - \theta_0)$, mass mr^2, and spring constant k. From this, the frequency of oscillation in θ is given by

$$\omega_\theta = \sqrt{\frac{k}{mr^2}} = \left\{\frac{1}{mr^2}\left(\frac{\partial^2 V(r,\theta)}{\partial\theta^2}\right)_{\theta_0} + \frac{1}{mr^2} \frac{\left(\sin^2\theta_0 + 3\cos^2\theta_0\right)}{\sin\theta_0 \cos\theta_0} \left(\frac{\partial V(r,\theta)}{\partial\theta}\right)_{\theta_0}\right\}^{1/2} .$$

NASA-Langley, 1969 —— 30 G-928

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546

OFFICIAL BUSINESS

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION

11C 001 55 51 3DS 69178 00903
AIR FORCE WEAPONS LABORATORY/AFWL/
KIRTLAND AIR FORCE BASE, NEW MEXICO 87117

ATT . LOU B... , ACTING CHIEF TECH. LI

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546